It Takes Two to Tango:

Estimation of the Zero-Risk Premium Strike of a Call Option via Joint Physical and Pricing Density Modeling

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Outline of the presentation

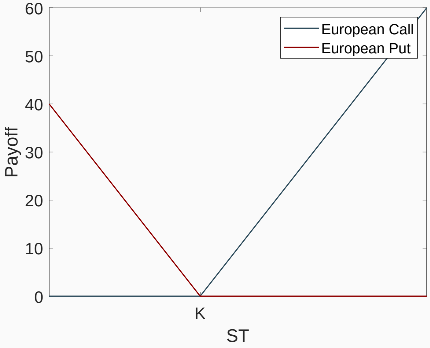
1. Introduction
2. From P to Q, and vice versa
3. (Zero-)Risk Premium
4. Data Example

Introduction

Introduction

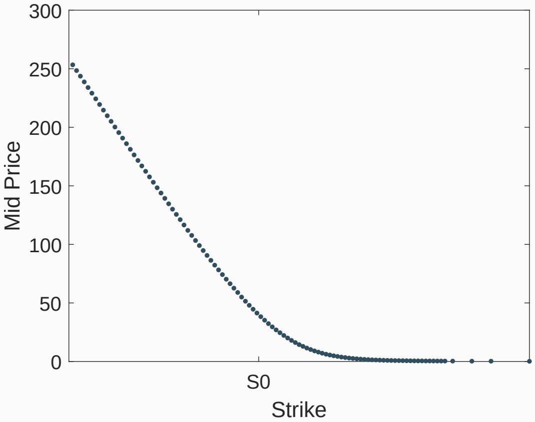
> A European **call** (**put**) option gives the right to the holder of the option to **buy** (**sell**), at a predetermined future time point T, called the maturity, the underlying asset S, for a predetermined price K, called the strike.

- Payoff European Call $= (S_T - K)^+$
- Payoff European Put $= (K - S_T)^+$



Introduction

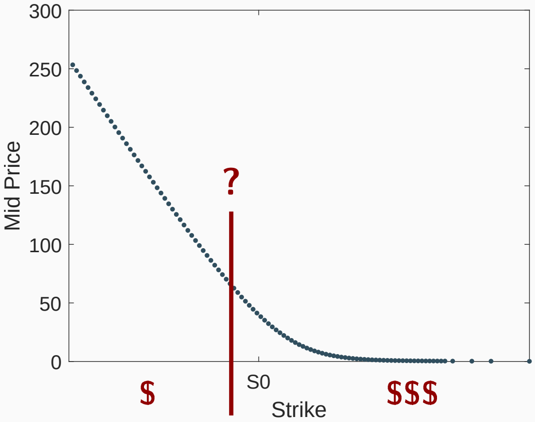
> Mid price of European **call** options on the S&P 500 index (SPX), on 15 March 2018, with a maturity of one month.



Introduction

Mid price of European **call** options on the S&P 500 index (SPX), on 15 March 2018, with a maturity of one month.

The **high-level objective** is to get a certain perception about the price.



Setting the scene

How **financial engineers** look at the world...

The $\mathcal{P}$**-world** is the physical world in which payoffs are realized:

$$\text{Expected Payoff European Call} = \mathbb{E}_{\mathcal{P}}[(S - K)^+].$$

The $\mathcal{Q}$**-world** is the pricing world, an artificial setting under which one determines the price:

$$\text{Price European Call} = \exp(-rT) \cdot \mathbb{E}_{\mathcal{Q}}[(S - K)^+].$$

Setting the scene

How do both worlds **compare** to each other?

$$\boxed{\text{Risk Premium} = \frac{\mathbb{E}_{\mathcal{P}}[\text{payoff}] - \mathbb{E}_{\mathcal{Q}}[\text{payoff}]}{\mathbb{E}_{\mathcal{Q}}[\text{payoff}]}}$$

- An **expensive** claim bears a negative risk premium.
- An **inexpensive** claim bears a positive risk premium.

Setting the scene

When looking at the risk premium, this paper focuses on the **interplay** between $\mathcal{P}$ and $\mathcal{Q}$.

To build a bridge...

...you need bricks.

- **Physical density** p of asset S, to calculate

$$\mathbb{E}_{\mathcal{P}}[\text{payoff}] = \int \text{payoff} \cdot p$$

- **Pricing density** g of asset S, to calculate

$$\mathbb{E}_{\mathcal{Q}}[\text{payoff}] = \int \text{payoff} \cdot g$$

From P to Q, and vice versa

Classical estimation of p and g

Physical density p

- Estimated based on historical data
  - backward looking
  - one new observation each day

Pricing density g

- Estimated based on option data
  - forward looking
  - multiple new observations each day
- Depending on an asset pricing model

  e.g. Black-Scholes, Variance Gamma...

Classical estimation of p and g

Physical density p → room for improvement

- Estimated based on historical data
  - backward looking
  - one new observation each day

alternative: simultaneous estimation of p and g

Pricing density g

- Estimated based on option data → rich source of information
  - forward looking
  - multiple new observations each day
- Depending on an asset pricing model

e.g. Black-Scholes, Variance Gamma...

Alternative physical density estimation - Step 1

Based on the method of Madan, Schoutens & Wang (2020).

Step 1: Pricing density as U-shaped perturbation of physical density

Classical asset **pricing theory**[1] states

$$\text{Price} = \exp(-rT)\mathbb{E}_{\mathcal{P}}[m \cdot \text{payoff}] = \exp(-rT)\int \text{payoff} \cdot m \cdot p,$$

with m a monotonically declining **pricing kernel**.

- Risk premia in European call options are always **positive**.
- Risk premia in European call options **increase** with strike.

[1] See Coval and Shumway (2001) and Cochrane (2005).

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empirically rejected

[1] See Coval and Shumway (2001) and Cochrane (2005).

Alternative physical density estimation - Step 1

Based on the method of Madan, Schoutens & Wang (2020).

Step 1: Pricing density as U-shaped perturbation of physical density

Recently, empirical evidence[2] is mounting that

$$\text{Price} = \exp(-rT)\mathbb{E}_{\mathcal{P}}[m \cdot \text{payoff}],$$

with m a **U-shaped pricing kernel**. We assume

$$g(x) = C \cdot ((1-\alpha) \cdot \exp(-\eta x) + \alpha \exp(\zeta x)) \cdot p(x),$$

with normalization constant C.

[2] See, e.g., Cuesdeanu and Jackwerth (2018), Sichert (2020) and Volkmann (2021).

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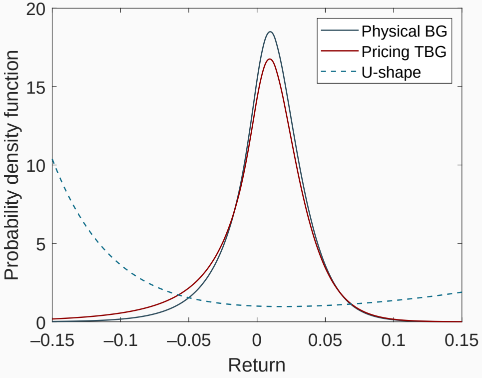

Alternative physical density estimation - Step 2

Based on the method of Madan, Schoutens & Wang (2020).

Step 2: Physical density follows a Bilateral Gamma model

Bilateral Gamma[3] models the asset in the $\mathcal{P}$-world as

$$\log(S_t) = \log(S_0) + b_p \cdot \gamma_p(c_p t) - b_n \cdot \gamma_n(c_n t),$$

where γ_p and γ_n are two independent standard Gamma processes.

[3] See Küchler & Tappe (2008).

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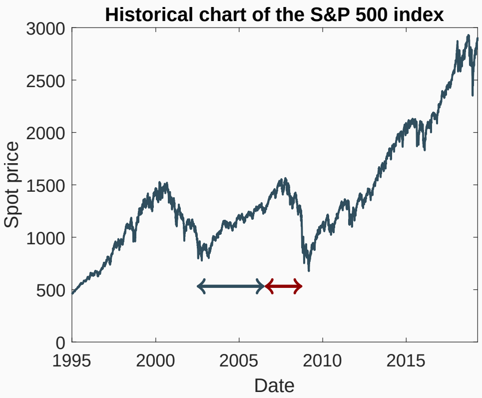


Alternative physical density estimation - Step 2

Based on the method of Madan, Schoutens & Wang (2020).

Step 2: Physical density follows a Bilateral Gamma model

$$\text{Physical density } p \xrightarrow{\cdot C \cdot \left((1-\alpha)\cdot e^{-\eta x} + \alpha \cdot e^{\zeta x}\right)} \text{Pricing density } g$$

Physical density p

- Bilateral Gamma
- Characterized by $[b_p, c_p, b_n, c_n]$

Pricing density g

- Tilted Bilateral Gamma
- Characterized by $[\eta, \zeta, \alpha, b_p, c_p, b_n, c_n]$

Alternative physical density estimation - Step 2

Based on the method of Madan, Schoutens & Wang (2020).

Step 2: Physical density follows a Bilateral Gamma model

$\cdot C \cdot \left((1-\alpha)\cdot \mathrm{e}^{-\eta x} + \alpha \cdot \mathrm{e}^{\zeta x}\right)$

Physical density p

- Bilateral Gamma
- Characterized by

$\boxed{[b_p, c_p, b_n, c_n]}$

from option data

Pricing density g

- Tilted Bilateral Gamma
- Characterized by

$\boxed{[\eta, \zeta, \alpha, b_p, c_p, b_n, c_n]}$

from option data

Alternative physical density estimation - Step 3

Step 3: Calibration of the Tilted Bilateral Gamma model

1. **Calibrate** the model parameters on option data.
   a. Calculate the model price of a call option with Carr-Madan[4] formula
   $$MoEC(K,T) = \frac{\exp(-\alpha \log(K))}{\pi} \int_0^\infty \exp(-\mathrm{i}\nu \log(K))\varrho(\nu)d\nu,$$
   where
   $$\varrho(\nu) = \frac{\exp(-rT)\mathbb{E}_{\mathcal{Q}}[\exp(\mathrm{i}(\nu-(\alpha+1)\mathrm{i})\log(S_T))]}{\alpha^2+\alpha-\nu^2+\mathrm{i}(2\alpha+1)\nu},$$
   $$= \frac{\exp(-rT)\phi^{\mathcal{Q}}_{\log(S_T)}(\nu-(\alpha+1)\mathrm{i};T)}{\alpha^2+\alpha-\nu^2+\mathrm{i}(2\alpha+1)\nu}.$$
   b. Minimize the distance between model prices and market prices.

[4] See Carr and Madan (1999).

Alternative physical density estimation - Step 3

Step 3: Calibration of the Tilted Bilateral Gamma model

2. Use an **inverse Fourier transform** to find the densities p and g.
   a. A closed-form expression exists for the characteristic functions ϕ_p and ϕ_g under the Bilateral and Tilted Bilateral Gamma model.
   b. We use the general relation

$$f(u) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(-iux)\phi(x)dx$$

to estimate the density functions.

(Zero-)Risk Premium

The zero-risk premium strike of an option

For a European call option, we have

$$\text{risk premium } EC(K,T) = \frac{\mathbb{E}_{\mathcal{P}}[(S_T - K)^+] - \mathbb{E}_{\mathcal{Q}}[(S_T - K)^+]}{\mathbb{E}_{\mathcal{Q}}[(S_T - K)^+]}.$$

For a fixed maturity T, we are interested in identifying the **zero-risk premium strike** K_T such that

$$\boxed{\mathbb{E}_{\mathcal{P}}[(S_T - K_T)^+] = \mathbb{E}_{\mathcal{Q}}[(S_T - K_T)^+].}$$

Existence of a zero-risk premium strike

In the paper, we show that, under the assumption of a **U-shaped pricing kernel**,

1. there **exists** a zero-risk premium strike for the European **call** option,
2. the zero-risk premium strike is **unique**, i.e., it indicates the transition point from which on call options can be considered expensive,
3. there **does not exist** a zero-risk premium strike for the European **put** option.

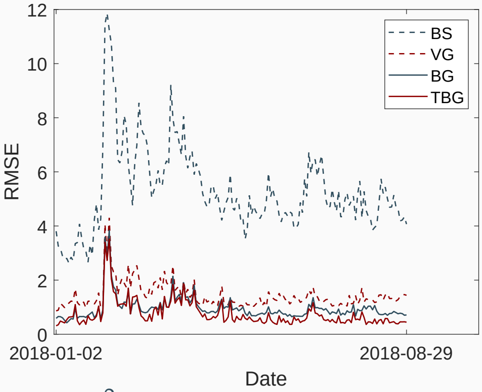
$\longrightarrow$ We focus on European call options.

Data Example

Data and Pricing performance

- Option data on the **S&P500 index**, between 2 January 2018 and 29 August 2018.

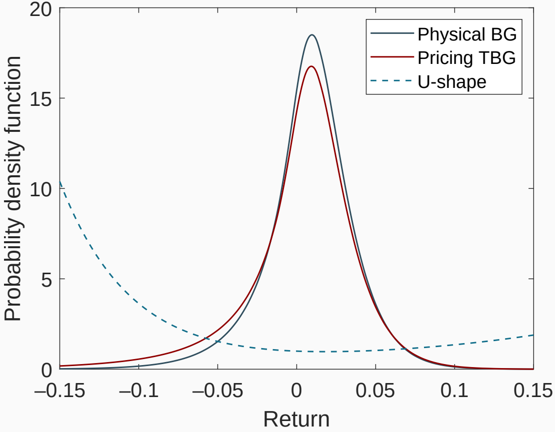
- Tilted Bilateral Gamma outperforms other models with respect to the calibration error:



$$\mathrm{RMSE}^2 = \frac{1}{N}\sum_{i=1}^{N}(\text{market price}_i - \text{model price}_i)^2.$$

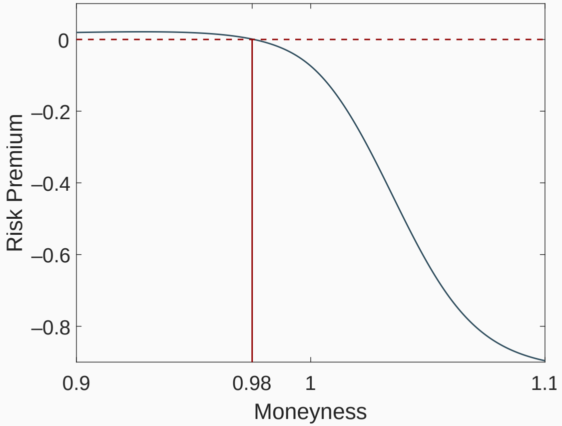
Physical and pricing density on 15 March 2018

- Probability density functions with a **maturity of one month**.
- The U-shape is more pronounced in the left tail.



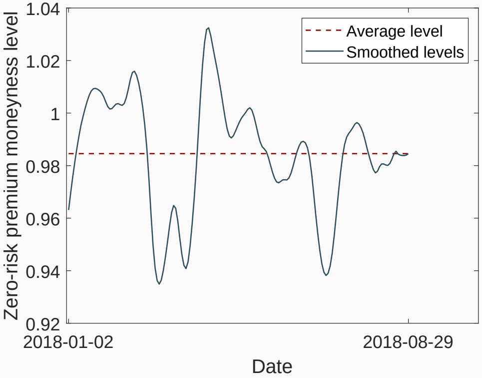
Risk premium on 15 March 2018

- The risk premium slightly increases before it decreases with strike.
- The **zero-risk** premium moneyness level amounts around **98%** of the spot price.



Evolution of the zero-risk premium strike

- The average zero-risk premium moneyness level amounts 98.5%.
- Zero-risk premium strikes are located in-the-money.
  - further away in-the-money call options are **inexpensive**
  - at-the-money and out-of-the-money call options are **expensive**
- Day-to-day fluctuations are rather small in absolute value.



Conclusion

- The **Tilted Bilateral Gamma model** makes it possible to simultaneously estimate both the physical and pricing density based on **option data** of the underlying asset.
- This provides us with all necessary information to estimate the **zero-risk premium** strike of a European call option.
  - Under the assumption of a U-shaped pricing kernel, this strike is unique; it indicates the transition point from which on call options can be considered expensive.
  - The data example shows that this strike is, on average, located slightly in-the-money for the S&P500 index.

Thank you!

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https://www.mdpi.com/2227-9091/9/11/196

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Additional information

Return - VIX - (ZRPS-1)